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THE PROPAGATION OF SOUND IN A FLUID UNDER AN ELASTIC PLATE WITH A CRACK†

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Diffraction by a short rectilinear crack in a thin elastic plate which is in a contact with a uniform acoustic half-space is investigated. The asymptotic forms of the radiation patterns of the scattered waves are constructed. The results obtained are compared with the scattering diagram of flexural waves at a crack in an isolated plate.

1. FORMULATION OF THE PROBLEM

LET A PLATE $\{z=0\}$ with a crack along the section $\Lambda=\{|x|<a,\ y=0,\ z=0\}$, the oscillations of which were investigated in [1], be in a contact with an acoustic medium. Suppose that the acoustic medium is uniform and located only on one side of the plate.

The oscillations of the system described are generated by a source which is not specified in detail. It is assumed that the sound pressure field u and the plate displacement field ξ_0, generated by the source in the problem for a plate without a crack, are known. It is required to determine the component of the wave process scattered by the crack.

The pressure in the medium satisfies the Helmholtz equation

$$\Delta u+k^2u=0,\quad z>0 \tag{1.1}$$

Here k is the wave number of sonic oscillations. The radiation conditions are satisfied at infinity. The boundary condition at the plate has the form

$$\begin{gathered}\{(\partial^2/\partial x^2+\partial^2/\partial y^2)^2-k_0^4\}\xi+D^{-1}u|_{z=0}=0,\quad \{x,y\}\notin\Lambda,\\ \xi=\rho^{-1}\omega^{-2}\partial u/\partial z|_{z=0},\quad k_0^4=h\rho\omega^2/D\end{gathered} \tag{1.2}$$

In this case, it is assumed that the plate performs flexural oscillations described by the function ξ. The wave number of the plate oscillations k is defined by the parameters D, ρ and h (the bending rigidity, the material density and the plate thickness, respectively), and by the frequency ω.

The conditions at the crack edges which imply the absence of a transverse force and a bending moment agree with the corresponding conditions in the problem for an isolated plate [1]

$$\begin{gathered}\mathbf{M}^{\pm}\xi\equiv\lim_{y\to\pm0}(\xi_{yy}+\sigma\xi_{xx})=0,\qquad |x|<a\\ \mathbf{F}^{\pm}\xi\equiv\lim_{y\to\pm0}(\xi_{yyy}+(2-\sigma)\xi_{xxy})=0,\quad |x|<a\end{gathered} \tag{1.3}$$

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Here σ is Poisson's ratio of the plate material.

For a sound pressure u in the neighbourhood of the crack the Meixner conditions are imposed

$$\nabla u = O(r^{-\delta}), \quad \delta < 1 \tag{1.4}$$

while the plate displacement near the crack ends must satisfy the condition that the energy is finite [2]

$$\nabla \xi = O(r^{\delta}), \quad \delta \geqslant 0 \tag{1.5}$$

By stretching the coordinates it may be assumed that $a=1$. For brevity, this is the case considered below.

2. INTEGRAL EQUATIONS

The problem is reduced to a system of integral equations over a segment. The scattered field u_1 will be determined using a Fourier transformation with respect to x and y as a sum of four terms

$$u_1 = -\frac{2i}{\pi} k_0^4 \sum_{j=0}^{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ik_0\lambda x} e^{ik_0\mu y} p_j(\lambda)\mu^j \frac{\exp(-L_1(\lambda,\mu)z)}{L(\lambda,\mu)} d\mu \tag{2.1}$$

$$L(\lambda,\mu) = -k_0^4 L_0(\lambda,\mu) L_1(\lambda,\mu) + \nu, \quad \nu = \rho\omega^2 / D$$

$$L_0(\lambda,\mu) = (\lambda^2 + \mu^2)^2 - 1, \quad L_1(\lambda,\mu) = \sqrt{k_0^2(\mu^2+\lambda^2) - k^2}$$

$p_j(\lambda)$ being unknown. The function $L(\mu, \gamma)$ is the Fourier transform of condition (1.2) at the plate.

The behaviour of the unknown functions $p_j(\lambda)$ at infinity are defined by conditions (1.5)

$$p_j(\lambda) = O(\lambda^{\delta - j}), \quad \delta > 0 \tag{2.2}$$

It can be established that in this case the Meixner conditions (1.4) are satisfied. The contour of integration circumvents from below the poles of the integrand that lie on the positive semi-axis, and from above those on the negative semi-axis. Such a choice of the contour follows from the radiation condition.

Representation (2.1) satisfies the Helmholtz equation and the radiation condition with respect to the z coordinate. The boundary condition (1.2) yields the integral equations

$$\int_{-\infty}^{+\infty} e^{ik_0 x\tau} p_j(\tau) d\tau = 0, \quad |x| > 1 \tag{2.3}$$

We now turn to the boundary-contact conditions (1.3). The plate displacement is expressed by the Fourier double integral

$$\xi_1 = -\frac{2ik_0^4}{\rho\omega^2\pi} \sum_{j=0}^{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ik_0\lambda x} e^{ik_0\mu y} p_j(\lambda)\mu^j \frac{L_1(\lambda,\mu)}{L(\lambda,\mu)} d\mu\, d\lambda \tag{2.4}$$

Passing to the limit with respect to y in the integrand, when substituting this expression into the boundary-contact conditions (1.3), we obtain a divergence of the integrals, which must be regularized. To simplify this procedure, we separate out the component in (2.4) corresponding to the vacuum problem. To do this, the fraction in the integrand in (2.4) is represented in the form

$$\frac{L_1(\lambda,\mu)}{L(\lambda,\mu)}=\frac{k_0^{-4}}{L_0(\lambda,\mu)}-\frac{\nu k_0^{-4}}{L_0(\lambda,\mu)L(\lambda,\mu)} \tag{2.5}$$

The first term in (2.5) produces the displacement $\xi_1^{(0)}$ in the problem for an isolated plate, as is readily demonstrated by evaluating the integrals with respect to λ. (Henceforth all the variables and parameters corresponding to the problem for an isolated plate will be marked with a zero superscript, the unity superscript refers to corrections.) The expressions obtained on substituting this term into the boundary-contact conditions were derived in [1].

We now turn to the correction term $\xi_1^{(1)}$ which corresponds to the second term in (2.5). For this term the decay of the integrand appears to be sufficient to perform differentiation and passage to the limit in the integrand. As a result, we obtain integral equations over a segment which, together with (2.3), complete the reduction of the initial problem to a system of the paired dual integral equations.

The integral equations that meet the requirements of $\mathbf{F}^+\xi=\mathbf{F}^-\xi$ and $\mathbf{M}^+\xi=\mathbf{M}^-\xi$ along the entire axis admit of an explicit solution

$$p_0(\mu)=\sigma\mu^2 p_2(\mu),\quad p_1(\mu)=(2-\sigma)\mu^2 p_3(\mu)$$

The remaining contact conditions reduce to the equations

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{ik_0x\mu}h_2(\mu)p_2(\mu)d\mu=\rho\omega^2\mathbf{M}\xi_0(x),\quad |x|<1\\ &\int_{-\infty}^{\infty} e^{ik_0x\mu}h_3(\mu)p_3(\mu)d\mu=\rho\omega^2\mathbf{F}\xi_0(x),\quad |x|<1\end{aligned} \tag{2.6}$$

The functions h_2 and h_3 differ from those reported in [1] by the additions generated by the correction terms (2.5)

$$h_n(\mu)=h_n^{(0)}+h_n^{(1)}$$

$$h_2^{(0)}(\mu)=\zeta_-(\mu)(1-\mu^2)^{-1/2}-\zeta_+(\mu)(1+\mu^2)^{-1/2}$$

$$h_3^{(0)}(\mu)=-i\zeta_+(\mu)(1-\mu^2)^{1/2}-\zeta_-(\mu)(1+\mu^2)^{1/2}$$

$$\zeta_\pm(\mu)=[(1-\sigma)\mu^2\pm 1]^2$$

$$h_2^{(1)}(\mu)=\frac{2i\nu}{\pi}\int_{-\infty}^{\infty}\frac{(\lambda^2+\sigma\mu^2)^2}{L_0(\lambda,\mu)L(\lambda,\mu)}d\lambda$$

$$h_3^{(1)}(\mu)=\frac{2i\nu}{\pi}\int_{-\infty}^{\infty}\frac{(\lambda^2+(2-\sigma)\mu^2)^2}{L_0(\lambda,\mu)L(\lambda,\mu)}\lambda^2 d\lambda$$

The further mathematical treatment is a repetition of that reported in [1]. To satisfy Eqs (2.3), we will seek $p_j(\mu)$ in the form

$$p_n(\mu)=\int_{-1}^{1}q_n(t)e^{-ik_0t\mu}dt$$

The behaviour of the functions $q_j(t)$ at the ends of the integration interval is described by the asymptotic forms (2.2)

$$q_j(t) = (1-t^2)^{j-2+\delta} Q(t), \quad \delta < 0, \quad Q \in C([-1,1]) \tag{2.7}$$

Changing the order of integration in (2.6) for $q_j(t)$ we obtain the integral equations of convolution over the section

$$\mathbf{H}_n q_n \equiv \int_{-1}^{1} H_n(x-t) q_n(t) dt = k_0^{2(1-n)} f_n(x), \quad |x| < 1 \tag{2.8}$$

The singular terms of the kernels H_n are identical with those calculated in [1] for the case of diffraction at a crack in an isolated plate. Therefore, integral equations (2.8) are uniquely solvable in the classes (2.7), the factor δ being equal to $\frac{1}{2}$. The solutions will be sought by the method of orthogonal polynomials

$$q_2(t) = (1-t^2)^{\frac{1}{2}} \sum_{l=0}^{\infty} \alpha_l U_l(t), \quad q_3(t) = (1-t^2)^{\frac{3}{2}} \sum_{l=0}^{\infty} \beta_l C_l^{(2)}(t)$$

The choice of Chebyshev polynomials of the second kind $U_l(t)$ and of Gegenbauer polynomials $C_l^{(2)}(t)$ is due to the fact that these polynomials from the eigenfunctions of the higher-terms of the integral operators H_2 and H_3.

As a result, the integral equations reduce to the following infinite algebraic systems

$$\begin{aligned} &-\kappa(1+1)\cdot\alpha_1 + \pi^{-2} k_0^2 \sum_{l=0}^{\infty} A_{lm}\alpha_m = \pi^{-2} f_1^{(2)} \\ &\frac{\kappa}{4}(1+1)^2(1+2)(1+3)^2\beta_1 + \pi^{-2} k_0^3 \sum_{l=0}^{\infty} B_{lm}\beta_m = \pi^{-2} f_1^{(3)} \\ &\kappa = (1-\sigma)(3+\sigma) \\ &f_1^{(2)} = i\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_l(x) \mathbf{M}\xi_0(x) dx \\ &f_1^{(3)} = -\int_{-1}^{1} (1-x^2)^{\frac{3}{2}} C_l^{(2)}(x) \mathbf{F}\xi_0(x) dx \end{aligned} \tag{2.9}$$

The decrease of $h_j^{(1)}(\mu)$ at infinity leads to the decrease of the indices 1 and m of the corrections to the matrices $A^{(0)}$ and $B^{(0)}$ which makes it possible, using the procedure similar to that of [1], to prove the unique solvability of systems (2.9).

Changing the variables of integration

$$\mu = \tau\cos\alpha, \quad \lambda = \tau\sin\alpha \tag{2.10}$$

the corrections $A^{(1)}$ and $B^{(1)}$ can be expressed by the double integrals

$$A_{lm}^{(1)} = 16 k_0^2 \nu (l+1)(m+1) i^{l-m+1} \int_0^{\infty} \frac{G_{l+1\,m+1}^{(1)}(k_0\tau)}{L(\tau)} \frac{\tau^3}{\tau^4 - 1} d\tau$$

$$B_{lm}^{(1)} = -4 k_0 \nu \frac{(l+3)!(m+3)!}{l!m!} i^{l-m+1} \int_0^{\infty} \frac{G_{l+2\,m+2}^{(2)}(k_0\tau)}{L(\tau)} \frac{\tau^3}{\tau^4 - 1} d\tau$$

$$G_{lm}^{(1)}(\chi) = \int_0^{\pi/2} J_l(\chi\cos\alpha) J_m(\chi\cos\alpha) \frac{(\sin^2\alpha + \sigma\cos^2\alpha)^2}{\cos^2\alpha} d\alpha$$

$$G_{lm}^{(2)}(\chi) = \int_0^{\pi/2} J_l(\chi\cos\alpha) J_m(\chi\cos\alpha) \frac{(\sin^2\alpha + (2-\sigma)\cos^2\alpha)^2}{\cos^4\alpha} \sin^2\alpha\, d\alpha$$

$$L(\tau) = -k_0^4(\tau^4 - 1)\sqrt{k_0^2\tau^2 - k^2} + \nu$$

3. ASYMPTOTIC FORMS OF THE FIELD AT LONG DISTANCES

Let us consider two modes of excitation.
1. A plane acoustic wave

$$u^{(i)} = \exp\{ik(x\cos\varphi_0 \sin\theta_0 + y\sin\varphi_0 \sin\theta_0 + z\cos\theta_0)\}$$

2. A plane surface wave

$$u^{(i)} = \exp\{i\tau_0(x\cos\varphi_0 + y\sin\varphi_0) - \sqrt{\tau_0^2 - k^2}\, z\}$$

Here τ_0 is the positive root of the dispersion equation

$$\sqrt{\tau^2 - k^2}\,(\tau^4 - k_0^4) - \nu = 0$$

that asymptotically tends to $\nu^{1/5}$ as $\omega \to 0$ [3].

In the first case, the geometrical part of the field u_0 is defined by the superposition of the plane wave $u^{(r)}$, incident on and reflected from the plate

$$u^{(r)} = R(\theta_0)\exp\{ik(x\cos\varphi_0 \sin\theta_0 + y\sin\varphi_0 \sin\theta_0 - z\cos\theta_0)\}$$

$$R(\theta_0) = R_- / R_+, \quad R_\pm = ik\cos\theta_0(k^4 \sin^4\theta_0 - k_0^4) \pm \nu$$

($(R(\theta_0)$ is the reflection coefficient).
In the second case, the geometrical part consists of an incident wave $u^{(1)}$ only.

Let us now investigate the asymptotic forms of the diffraction part of the field in the far zone. To do this we change the variables (2.10) and evaluate the integral with respect to α. The diffraction part of the field consists of an expanding surface wave formed by the contributions of the residue at the pole $\tau = \tau_0 k_0^{-1}$

$$u_{surf} \approx \sqrt{\frac{2\pi}{\tau_0 r}}\, e^{i\tau_0 r - i\pi/4} e^{-\sqrt{\tau_0^2 - k^2}\, z}\, \Psi_0(\varphi)$$

and of the spherical wave which corresponds to the contribution of the saddle point $\tau = \tau_0 k_0^{-1} \cos\varphi$

$$u_{spher} \approx \frac{2\pi}{kR} e^{ikR - i\pi/2}\, \Psi_s(\varphi, \theta)$$

Here

$$\Psi_0(\varphi) = 2i(5\pi\tau_0)^{-1}\{p_2(\tau_0\cos\varphi)Q(\varphi) + p_3(\tau_0\cos\varphi)\tau_0\cos\varphi(\sin^2\varphi + (2-\sigma)\cos^2\varphi)\}$$

$$\Psi_s(\varphi,\theta) = \frac{-2\pi^{-1}k^4\cos\theta\sin^2\theta}{k\cos\theta(k^4\sin^2\theta - k^4) - \nu i}\{p_2(k\sin\theta\cos\varphi)Q(\varphi) +$$

$$+ p_3(k\sin\theta\cos\varphi)k\sin\theta\cos\varphi(\sin^2\varphi + (2-\sigma)\cos^2\varphi)\}$$

$$Q(\varphi) = \sin^2\varphi + \sigma\cos^2\varphi$$

Let us find the asymptotic forms of Ψ_0 and Ψ_s with respect to the small parameter k_0 (i.e. with respect to $k_0 a$ in the initial notation). To do this we consider the first equation of (2.9). It can be verified that the higher-order terms of the asymptotic forms of the diagrams are governed by the coefficient α_0. The other coefficients are of higher-order infinitesimal values

$$\alpha_0\{-\kappa+\pi^{-2}k_0^2(A_{00}^{(0)}+A_{00}^{(1)})\}+\ldots=i\pi^{-1}\mathrm{M}\xi_0(0,0) \qquad (3.1)$$
$$A_{00}^{(0)}+A_{00}^{(1)}=O(k_0^{-2}\nu^{2/5})$$

Here ξ_0 is the plate displacement corresponding to u_0. The off-diagonal terms that are denoted by dots may be neglected in view of the higher-order infinitesimal values of the coefficients α_2, α_4, ...

In the case of an incident plane acoustic wave, the right-hand side of Eq. (3.1) has the form

$$\pi^{-1}(1-R(\theta_0))k^3\cos\theta_0\sin^2\theta_0\,Q(\varphi_0)$$

which yields the asymptotic forms

$$\Psi_0^{(1)}(\varphi)=-2(5\pi\kappa)^{-1}k^3a^2\nu^{-1/5}\cos\theta_0\sin^2\theta_0\,Q(\varphi)Q(\varphi_0)$$
$$\Psi_s^{(1)}(\varphi,\theta)=ik^7a^2(\pi\kappa\nu)^{-1}\cos\theta_0\sin^2\theta_0\,Q(\varphi_0)\cos\theta\sin^2\theta\,Q(\varphi) \qquad (3.2)$$

For the second case of excitation, the right-hand side of (3.1) has the following form

$$i\pi^{-1}\sqrt{\tau_0^2-k^2}\,\tau_0^2Q(\varphi_0)$$

which yields

$$\Psi_0^{(2)}(\varphi)=-2i(5\pi\kappa)^{-1}\tau_0^3a^2\nu^{-1/5}Q(\varphi_0)Q(\varphi)$$
$$\Psi_s^{(2)}(\varphi,\theta)=-k^7\tau_0^3a^2(\pi\kappa\nu)^{-1}Q(\varphi_0)\cos\theta\sin^2\theta\,Q(\varphi) \qquad (3.3)$$

In formulae (3.2) and (3.3) we have used dimensional quantities again, and a is the half-length of the crack.

It is worth noting that the discrepancy between the values of the diagrams $\Psi_0^{(1)}(\varphi_0,\ \theta_0,\ \varphi)$ and $\Psi_s^{(2)}(\varphi,\ \varphi_0,\ \theta_0)$ is due to the fact that the plane acoustic and surface waves have different energies.

Comparing the asymptotic form (3.2) and the diagram in the problem of the scattering of the flexural waves by a crack in an isolated plate [1]

$$\Psi(\varphi_0,\varphi)=i(4\kappa)^{-1}(k_0a)^2Q(\varphi_0)Q(\varphi) \qquad (3.4)$$

it can be seen that the presence of an acoustic medium results in an enlargement of the diagram. In the case of an infinite crack [3], the effect will be the opposite. In an isolated plate the crack will completely reflect the incident flexural wave, and the presence of an acoustic wave results in the appearance of a transmitted flexural wave. Thus, the acoustic medium, on one hand, results in the appearance of additional "paths" of circumventing the obstacle and thereby decreases the effective scattering cross-section. On the other hand, due to the presence of an acoustic medium, an additional scattering channel (of a wave entering an acoustic medium) occurs, and the scattering cross-section increases. When the crack is long enough, it is obvious that the first effect predominates, which also explains the result obtained in [3]. For short cracks, as follows from a comparison of (3.3) and (3.4), the effect of the additionally occurring scattering channel predominates.

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